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Title: Banach Tarski Paradox and Amenability

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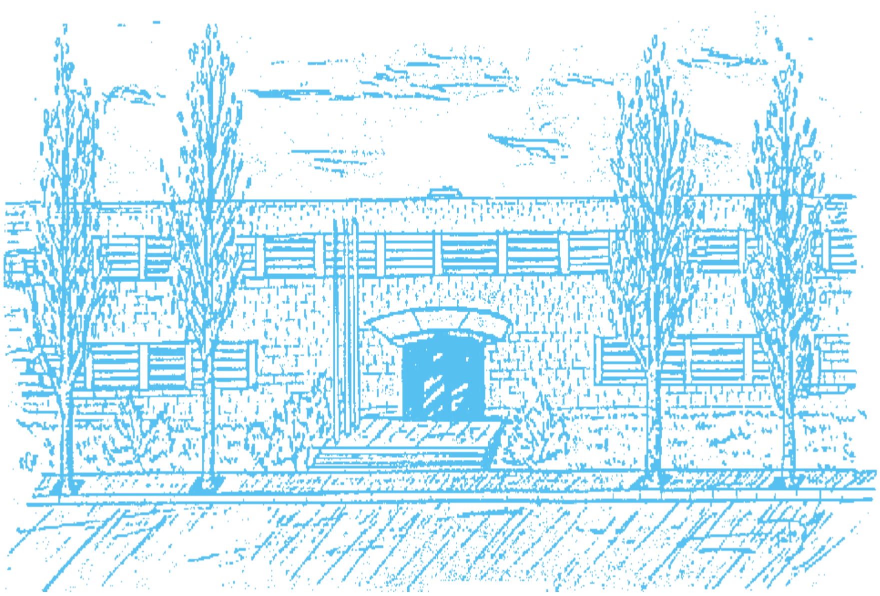

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Bachelor Thesis

Banach Tarski Paradox and Amenability

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To those who supported me and helped me grow up along the way.

Abstract

Keywords: Invariant measures, Amenability

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The main objective of this bachelor's thesis is to prove Banach-Tarski theorem. The theorem states that a ball in a 3-dimensional space can be split into finitely many pieces that can be rearranged to form two balls, each of the same size as the first one. The concept of amenability, which underlies the paradox, will be explained and characterized as well. We will also classify some groups in terms of amenability. Proving that groups in certain classes are all amenable and those in others classes are not is the approach that we will take to address this issue.

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Contents

Chapter 1
Introduction

The main goal in this project is to prove the Banach-Tarski paradox. The Banach-Tarski theorem states that we can take the unit ball in a space, split it into finitely many pieces, apply rigid motions to these pieces to form two balls, each with the same volume as the initial one. This fact is counterintuitive to our geometric and spatial intuition. What makes it shocking is the geometrical aspect of the statement, although this phenomenon also appears in the bijection between the odd or even numbers with the set of all integer numbers. Thus, being infinite is necessary for things to be paradoxical. Galileo noticed that perfect squares can be put into a one-to-one correspondence with the set of positive integers, even though the set of all integers seems more numerous than the set of perfect squares. Because of this, he realized that "the attributes 'equal', 'greater' and 'less' are not applicable to infinite... quantities" [**1**, p. 3].

Later on, in the nineteenth century, Georg Cantor (1845-1918) showed that infinite quantities can actually be compared. First, he looked at finite sets, pointing out that two finite sets would have the same cardinality if it was possible to define a one-to-one correspondence among their elements. He did the same with infinite sets. For example, the counting numbers would have the same cardinality as the even numbers since there is a clear one-to-one correspondence between them. Revolutionary concepts of Cantor's set theory were controversial. Henri Poincaré (1854-1912) would eventually refer to Cantor's set theory as "a malady, a perverse illness from which some day mathematics would be cured" [**3**, pp. 5-16]. In 1908, set theory was axiomatized by the German mathematician Ernst Zermelo (1871-1953) and in 1922 the logician Abraham Fraenkel (1891-1965) made additional contributions. This is why the system of eight to ten axioms, depending on the formulation, is today known as Zermelo-Fraenkel set theory. The most relevant statement for this project is the **Axiom of Choice**. It was and it is so controversial that still some mathematicians do not accept it. Those who accept it would refer to the ZFC as the Zermelo-Fraenkel set theory with the Axiom of Choice, and those who do not accept it would use the ZF set theory.

In 1914, Felix Haussdorf (1868-1942) published an article in which he asserted that a sphere minus countably many points can be partitioned into three disjoint subsets A, B and C in such a manner that A, B, C and $B \cup C$ are all congruent to each other. He proved this relying on the Axiom of Choice. Haussdorf was criticized for this result and some mathematicians saw it as an evidence to reject the Axiom of Choice [**3**, pp. 21-22]. Haussdorf, on the other hand, suggested that there was no paradox at all since certain sets of points of the sphere could not have definable surface measure, which again, was a controversial statement itself.

Ten years later, Stefan Banach (1892-1945) and Alfred Tarski (1902-1983) discovered independently a way to extend the paradox to the entire sphere. After this result, they started working together and in 1924 they published the paper *Sur la décomposition des ensembles de points en parties respectivement congruentes* [**4**] where the paradox was extended to the whole ball. They wrote it in French since it was conceived more as an international language for scientific publications than Polish. The Banach-Tarski paradox or Banach-Tarski theorem states that a ball can be decomposed into finitely many pieces and reassembled to form two balls, each identical in size to the original. This is the *duplication version* of the theorem. There is another version, known as the *strong form* or *magnification version* that asserts that a solid object in the space with any shape and size, for example a pea, can be split into finitely many pieces and reassembled to form a solid object of any other shape and size, say that of the sun. It is also known as the *pea and the sun version* [**3**, p. 23].

Since the initial ball is measurable, rigid motions preserve the distance and the volume and the final two balls are measurable as well, the pieces we split the ball into must be non Lebesgue measurable. It is precisely because of this paradox that not all the subsets in the space are Lebesgue measurable. The existence of this kind of paradoxes in certain spaces is closely related to the possibility of defining a finitely additive measure, such that it is invariant under rigid motions (or under the action of a group, depending on the set we are considering), defined in the set of all subsets and giving a specific set a finite measure. This paradox is possible in the space, so we will conclude that a measure with those properties can not be defined. The well known Lebesgue measure is invariant under the action of the isometry group and gives to the unit ball a finite measure and is finitely additive. In fact it is not only finitely additive, but countably additive. Is that a contradiction with the statement asserting that such a measure can not exist? No, because it is not defined in all the subsets, but only in those well known sets called Borel sets.

Because of the possibility to decompose a set into two families of sets that, if rearranged properly, can form two sets as the initial one, we call these kind of sets paradoxical sets. We also say they bear a paradoxical decomposition. Along the proof of the paradox, we realize that the property of being paradoxical is not intrinsic of a set. In fact, we will refer to sets being paradoxical regarding the action of a group. In the proof, using certain techniques and notation, we will transfer the 'paradoxability' of the isometry group into the space where it acts. What follows is to explore which groups are paradoxical and why, so the concept of amenability becomes crucial.

Those groups in which the paradox is not possible are called amenable. As we have said above, that is deeply related to the possibility of defining a measure with certain characteristics. In addition of being non paradoxical, or bearing such a measure, we will discuss another characterization of amenability regarding Følner's Condition. All three are equivalent and we will see some of the proofs of this equivalence. The others will be omitted in order to avoid extending the dissertation and because they use techniques this project does not include.

We will prove that the paradox is not possible in the plane using this concept. We will see that the isometry group in the plane is amenable, so there exists a measure with those characteristics. From the existence of this measure, even if we cannot explicitly define it, we can conclude that there will be no paradoxical sets with nonempty interior, hence the paradox is not possible.

In chapter 2 we will define what paradoxical sets and groups are. As an example we will see that a free group of rank 2 is paradoxical. We will introduce the concept of equidecomposability. This definition will be useful in chapter 4 when proving the paradox. It defines a equivalence relation that will allow us to see that some sets are paradoxical if we have already proved that another set in the same equivalence class is paradoxical.

In the next chapter we will talk about measures. We will see the equivalence for a set between being paradoxical and bearing a measure with certain properties. We will define amenability with those two concepts. It will be shown that if a paradox exists, such a measure cannot. Finally, we will define a left-invariant mean and we will see that bearing such a mean is equivalent to be amenable.

Chapter 4 develops one of the key objectives of the project. We will follow several steps to prove the paradox. Firstly, we will prove that there exist two rotations in the isometry group of the space that generate a free group. Secondaly, we will see that the action of this free group on the sphere minus countably many points will allow us to transfer the paradoxability of the free group into the sphere minus those points. Then, when proving that the sphere without this set of points is paradoxical we will need the Axiom of Choice, since we need to choose a representative of uncountably many orbits. Finally, using equidecompositions as defined previously we will extend this paradox from the sphere minus that set of points to the whole sphere, and after that, to the ball.

Chapter 5 provides another characterization for amenability: Følner's Condition. We will demonstrate that bearing a measure, such as those in chapter 3, will lead to the possibility of constructing a sequence of sets called Følner's sequence. We will be able to construct those sets in amenable groups and they will have a relatively small border comparing to the whole set. We will show that $\mathbb{Z}^n$ will bear those sets for any n. We will also see that it is impossible to define a family of sets like that in a free group. Finally we will show the equivalence between bearing a Følner's sequence and meeting Følner's Condition. We will prove that satisfying Følner's Condition implies being amenable, but we will not be able to prove the conversely.

In chapter 6 summarizes what has been discussed about amenability. We will prove some important results in order to classify groups as amenable. It will be proved that abelian groups are amenable and that subgroups of amenable groups are amenable. We will see that a group is amenable if and only if all its finitely generated subgroups are. We will study normal subgroups and quotients of amenable groups. With these results we will conclude that solvable groups and elementary groups are amenable. At this point we will observe that the class EG (of elementary groups) is inside the class AG (of amenable groups) which is inside the class NF (of groups not containing a free group). As a consequence of these results, the following chapter will briefly show that the isometry group of $\mathbb{R}^2$ is amenable, so the paradox cannot happen in the plane.

Chapter 8 will give a different perspective about amenability, displaying its relation with growth of groups. A new class of groups will be defined: those with intermediate growth. We will see that by knowing the growth rate of a group, we might know whether the group is amenable or not.

In order to complement this project, further studies should introduce some concepts about topology with the aim of proving those equivalences that have not been proved in the present work. Despite the large number of groups that have been classified in this work, it would also be interesting to study amenable yet not elementary groups and those that are not amenable but do not contain a free group.

Chapter 2
Paradoxical groups

Definition 2.1. Let G be a group acting on a set X and suppose $E \subseteq X$. E is G-**paradoxical** if for some positive integers m and n, there are pairwise disjoint subsets A_1, ... ,A_n, B_1, ... , B_m of E and g_1, ... , g_n, h_1, ... , $h_m \in G$ such that

$$E = \bigcup_{i}^{n} g_i A_i = \bigcup_{j}^{m} h_j B_j$$

The free group F with generating set M is the group of all finite reduced words using letters from $\{\sigma\ ,\ \sigma^{-1} | \sigma \in M\}$. The operation in the group is the concatenation, and after concatenating two words, we will eliminate every element $\sigma\sigma^{-1}$ and $\sigma^{-1}\sigma$. Any two free generating sets for a free group have the same size, which is called the rank of the free group. Free groups of the same rank are isomorphic.

Theorem 2.2. *A free group F of rank 2 is F-paradoxical, where F acts on itself by left multiplication.*

Proof. Suppose σ, τ are free generators of F. If ρ is one of $\sigma^{\pm 1}$, $\tau^{\pm 1}$. Let $[\rho]$ be the set of elements of F whose representation as a word in σ, σ^{-1}, τ, τ^{-1} begins, on the left, with ρ. Then $F = \{1\} \cup [\sigma] \cup [\sigma^{-1}] \cup [\tau] \cup [\tau^{-1}]$, and these subsets are pairwise disjoint. Furthermore, $[\sigma] \cup \sigma[\sigma^{-1}] = F$ and $[\tau] \cup \tau[\tau^{-1}] = F$. For if $h \in F \backslash [\sigma]$, then $\sigma^{-1}h \in [\sigma^{-1}]$ and $h = \sigma(\sigma^{-1}h) \in \sigma[\sigma^{-1}]$. $\square$

Any group acts naturally on itself by left translation. If a group G is paradoxical under its own action, we will say it is a **paradoxical group**.

PROPOSITION 2.3. *Let G be a paradoxical group that acts on a set X. We will say it acts on X without nontrivial fixed points, if for any $w \in G$, $w \neq id$ and $a \in X$, $w(a) \neq a$. Then X is G-paradoxical.*

PROOF. Let $\{A_i\}$, $\{B_j\}$, $\{g_i\}$, $\{h_j\}$ be the paradoxical decomposition of G. We can split the set X into orbits and, using the Axiom of Choice, choose a representative of each orbit. Let M be the set of representatives. We will call C_i the orbit of M by the elements in A_i and D_j the orbit of M by the elements in B_j:

$$C_i = \bigcup \{aw | a \in A_i, w \in M\}$$

$$D_j = \bigcup \{bw | b \in B_j, w \in M\}$$

Since $G = \cup_i^n g_i A_i$,

$$\bigcup_i^n g_i C_i = \bigcup_i^n \bigcup \{g_i aw | a \in A_i, w \in M\} = \bigcup_i^n g_i A_i M = GM = X$$

We have been able to transfer this paradoxical decomposition because since there are not fixed points, there exists an isomorphism between the paradoxical group G and the set X. $\square$

Since a subgroup of a group acts by left translation on the whole group, without non trivial fixed points, the following result is an immediate consequence of the proposition.

COROLLARY. *A group with a paradoxical subgroup is paradoxical. Hence any group with a free subgroup of rank 2 is paradoxical.*

It is interesting to define the following concept [**1**, p.23], since it will help to rewrite the property of being paradoxical regarding the action of a group.

DEFINITION 2.4. Suppose G acts on X and A, $B \subseteq X$. Then A and B are G-equidecomposable if A and B can each be partitioned into the same finite number of respectively G-congruent pieces. Formally,

$$A = \bigcup_{i=1}^{n} A_i \quad B = \bigcup_{i=1}^{n} B_i$$

$A_i \cap A_j = \emptyset = B_i \cap B_j$ if $i < j < n$ and there are $g_1, \ldots$, $g_n \in G$ such that, for each $i \leq n$, $g_i(A_i) = B_i$.

The notation $A \sim_G B$ will be used to denote the equidecomposability relation. G will be suppressed when X is a metric space and G is the full isometry group, or when the group G meant is obvious. It is easy to see that $\sim_G$ is an equivalence relation, and so, transitive. If $A \sim_G B$ and the minimum number of pieces to do the equidecomposability is n, and $B \sim_G C$ with a least m pieces, then $A \sim_G C$ would need at most nm pieces.

PROPOSITION 2.5. *Let E_1 and E_2 be two sets. If they are G-equidecomposable and E_1 is G-paradoxical, then E_2 is G-paradoxical as well.*

PROOF. If E_1 and E_2 are G-equidecomposable, then there exists $\{A_i\}$ such that $E_1 = A_1 \cup \ldots \cup A_n$ and there exists $\{f_i\}$ such that $E_2 = f_1A_1 \cup \ldots \cup f_nA_n$. E_1 is G-paradoxical, so there exists $\{B_j\}, \{C_k\}$ and there exists $\{g_j\}, \{h_k\}$ such that

$$E_1 = g_1B_1 \cup \ldots \cup g_mB_m = h_1C_1 \cup \ldots \cup h_lC_l$$

Let's call $B_{ij} = A_i \cap B_j$ and $C_{ik} = A_i \cap C_k$.

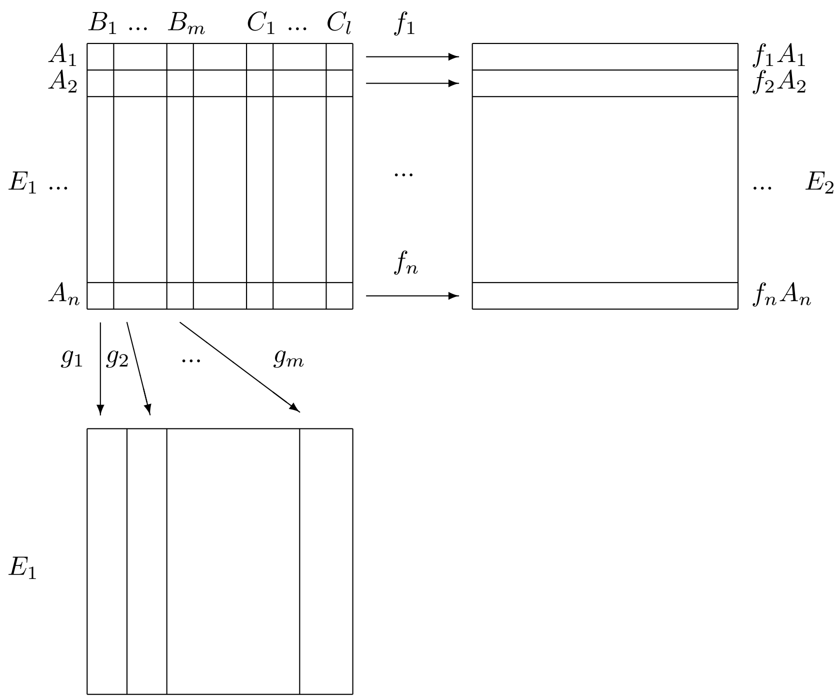
$$\bigcup_{i=1}^{n} B_{ij} = B_j \quad \text{and} \quad \bigcup_{i_1}^{n} C_{ik} = C_k$$

So

$$E_1 = g_1 B_{11} \cup g_1 B_{12} \cup ... \cup g_1 B_{1n} \cup g_2 B_{21} \cup ... \cup g_m B_{mn} =$$

$$= h_1 C_{11} \cup h_1 C_{12} \cup ... \cup h_1 C_{1n} \cup h_2 C_{21} \cup ... \cup h_l C_{ln}$$

This is easy to see in a graphic:



Now, we will apply the corresponding f_i to send these sets from E_1 to E_2:

$$\bigcup_{i,j}^{n,m} f_i g_j f_i^{-1}(f_i B_{ij}) = \bigcup_{i,j}^{n,m} f_i g_j (B_{ij}) = E_2$$

$$\bigcup_{i,k}^{n,l} f_i h_k f_i^{-1}(f_i C_{ik}) = \bigcup_{i,k}^{n,l} f_i h_k (C_{ik}) = E_2$$

We have the paradoxical decomposition of E_2. The explicit sets of the paradoxical decomposition are $f_i B_{ij}$ and $f_i C_{ik}$. The elements of the group would be $f_i g_j f_i^{-1}$ and $f_i h_k f_i^{-1}$. Since B_{ij} and C_{ik} are disjoint, $f_i B_{ij}$ and $f_i C_{ik}$ are disjoint as well. □

Now we are able to rephrase the definition of G-paradoxical.

LEMMA 2.6. *Let G be a group acting on a set X, and $E \subseteq X$ be a subset. E is G-paradoxical if and only if E contains disjoint sets A, B such that $A \sim_G E$ and $B \sim_G E$.*

With such a definition and because of the transitivity of the equivalence relation, it is easy to see that being G-paradoxical is a property of a whole class of equivalence. So,

PROPOSITION 2.7. *Let G act on X and E, $E' \subseteq X$ be G-equidecomposable subsets, $E \sim_G E'$. E is paradoxical if and only if E' is.*

Chapter 3
Measures in groups

To easily see the relation between the property of being paradoxical and of bearing a finitely additive, left-invariant measure ($\mu(gA) = \mu(A)$ for $g \in G$, $A \subseteq G$) on $\mathcal{P}(G)$ giving a finite value to G, we need the next definition and proposition [**1**, p. 18]:

Definition 3.1. Let G be a group acting on a set X and let $E \subseteq X$ be a subset. We will say E is μ-negligible if $\mu(E) = 0$, whenever μ is a finitely additive, G-invariant measure on $\mathcal{P}(X)$ with $\mu(X) < \infty$.

Proposition 3.2. *If E is G-paradoxical, then E is μ-negligible.*

Proof. Suppose μ is a finitely additive, G-invariant measure on $\mathcal{P}(X)$ with $\mu(E) < \infty$. Since E is G-paradoxical, there exist pairwise disjoint subsets $\{A_i\}$, $\{B_j\}$ and elements of the group $\{g_i\}$, $\{h_j\}$ such that

$$E = \bigcup_{i}^{n} g_i A_i = \bigcup_{j}^{m} h_j B_j$$

With such a measure,

$$\mu(E) \geq \sum_{i}^{n} \mu(A_i) + \sum_{j}^{m} \mu(B_j) = \sum_{i}^{n} \mu(g_i A_i) + \sum_{j}^{m} \mu(h_j B_j) \geq$$

$$\geq \mu(\bigcup_{i}^{n} g_i A_i) + \mu(\bigcup_{j}^{m} h_j B_j) = \mu(E) + \mu(E) = 2 \cdot \mu(E)$$

Since $\mu(E) < \infty$, this means $\mu(E) = 0$. $\square$

So we have seen that if paradoxical sets exist, these must be negligible. So, in a group G with G-paradoxical sets, we could define a finitely additive, G-invariant measure defined on $\mathcal{P}(G)$, but this measure could not give any value to these G-paradoxical sets but zero. That is exactly what Tarski theorem states. In fact, non-paradoxical groups coincide with the groups that bear a left-invariant, finitely additive measure of total measure one that is defined on all subsets. Such groups are called **amenable**. This leads to the following theorem [**1**, p. 128]

THEOREM 3.3. *(Tarski theorem) Let G be a group acting on a set X and $E \subseteq X$ be a subset. Then there is a finitely additive, G-invariant measure $\mu : \mathcal{P}(X) \to [0, \infty]$ with $\mu(E) = 1$ if and only if E is not G-paradoxical.*

We have proved that a paradoxical group can not bear such a measure, so, a group that bears it, will necessarily be non paradoxical. The conversely will not be proved in this project since the proof [**2**, p. 8-11] uses topological concepts that, if included, would extend too much this project.

DEFINITION 3.4. Let G be a group and let μ be a finitely additive measure on $\mathcal{P}(G)$ such that $\mu(G) = 1$ and is left-invariant . We will call μ a measure on G. An **amenable group** is one that bears such a measure and AG denotes the class of all amenable groups.

This definition of amenability [**1**, p. 146] establishes whether a group is amenable or not, depending exclusively on the existence of such a measure. We might be interested in defining a specific measure whenever it is possible, but sometimes it will be impossible to define it and we will have to prove the existence.

If G is an infinite group there cannot be a left-invariant measure μ on $\mathcal{P}(G)$ such that $\mu(G) = 1$ and is countably additive.

For a group G, let $B(G)$ denote the collection of bounded real-valued functions on G. Let μ be a measure on G. We can construct an integral from this measure with the triple $(G, \mathcal{P}(G), \mu)$. First, defining the integral on simple functions, and then, via limits, on all measurable functions. Here, all sets are measurable, since μ is defined in $\mathcal{P}(G)$, and hence all functions. So, $\int f d\mu$ defines a linear functional on all of $B(G)$. This means that $\int f d\mu : B(G) \to \mathbb{R}$ such that

$$\int (\alpha f + \beta g) d\mu = \alpha \int f d\mu + \beta \int g d\mu$$
$$\text{for} \quad f,g \in B(G) \quad \text{and} \quad \alpha,\beta \in \mathbb{R}$$

Standard theorems such as the monotone and dominated convergence cannot be proved because μ is not necessarily countably additive. A real number $\int f d\mu$ is assigned to each $f \in B(G)$ [**1**, p. 147], and this integral satisfies these properties:

Definition 3.5. Let G be a group and μ be a finitely additive, left-invariant measure on $\mathcal{P}(G)$ such that $\mu(G) = 1$. A linear functional $\int f d\mu$ is called a **left-invariant mean** on $B(G)$ if it satisfies the following properties:

(i) $\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu$, where a, $b \in \mathbb{R}$.
(ii) $inf\{f(g) : g \in G\} \leq \int f\mu \leq sup\{f(g) : g \in G\}$
(iii) For each $g \in G$, $f \in B(G)$, $\int ({}_g f) d\mu = \int f d\mu$, where $({}_g f)(h) = f(g^{-1}h)$; that is, the integral is left-invariant.

Therefore, an amenable group always bears a left-invariant mean. Conversely, if $F : B(G) \to \mathbb{R}$ is a left-invariant mean, then we can define a measure in the group $\mu(A) = F(\chi_A)$. Hence, PROPOSITION 3.6. *A group is amenable if and only if bears a left-invariant mean.*

Chapter 4
Banach-Tarski Paradox in $\mathbb{R}^3$

The Banach-Tarski paradox states that we can split up a ball in the space into finitely many pieces and, after applying rigid motions to those pieces, rearrange them forming two balls, each one with the volume of the first one.

With the definition of a paradoxical set, it is easier to enunciate the Banach-Tarski paradox, assuming that G_n (isometry group) acts on $\mathbb{R}^n$.

THEOREM 4.1. *(Banach-Tarski paradox) $B^3 \subset \mathbb{R}^3$ is G_3-paradoxical.*

We have already seen this in the previous chapter, but now we will see it in the specific case with $\mathbb{R}^3$ playing the role of the set X and Lebesgue measure playing the role of μ. Let V be the volume of B^3. If A_i and B_j are a paradoxical decomposition of B^3, then

$$V \leq \sum_{i=1}^{n} \mu(g_i A_i) = \sum_{i=1}^{n} \mu(A_i)$$

$$V \leq \sum_{j=1}^{m} \mu(h_j B_j) = \sum_{j=1}^{m} \mu(B_j)$$

but then

$$2V \leq \sum_{i=1}^{n} \mu(A_i) + \sum_{j=1}^{m} \mu(B_i) \leq V$$

So, we can conclude that $\{A_i\}_{i=1,2,...,n}$ and $\{B_j\}_{j=1,2,...,m}$ must be non-Lebesgue measurable sets.

PROPOSITION 4.2. *Let $SO(3)$ be the group of rotations in $\mathbb{R}^3$. It has a subgroup isomorphic to F_2.*

PROOF. The elements of $SO(3)$ can be represented with matrices $3x3$ since there exist a bijection between these groups. Let α and β be two rotations with angle ϕ such that $cos(\phi) = 1/3$ and let their axis intersecting at an angle of $90°$. For example:

$$\alpha = \begin{pmatrix} \frac{1}{3} & -\frac{2\cdot\sqrt{2}}{3} & 0 \\ \frac{2\cdot\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\cdot\sqrt{2}}{3} \\ 0 & \frac{2\cdot\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

Let's see that the group L generated by α and β is free, with $\{\alpha, \beta\}$ a basis.Let w be any word in α and β. If w ends by β, conjugate it by the appropriate power of β so it ends with $\alpha^{\pm 1}$. Let β^n be that powder, w is the identity if and only if $\beta^{-n} w \beta^n$ is the identity. So we can assume that the word w ends with $\alpha^{\pm}$.

Now, let's apply any word starting by α to the vector $(1, 0, 0)$. We claim that $w(1, 0, 0)$ has the form $(a, b, \sqrt{2}, c)/3^k$ where a, b, c are integers and b is not a multiple of 3. If this is true, then $w(1, 0, 0) \neq (1, 0, 0)$, which is the required contradiction. The claim will be proved by induction on the length of w. If the length of w is one, then $w = \alpha^{\pm}$ and $w(1, 0, 0) = (1, \pm 2\sqrt{2}, 0)/3$. Now, we can suppose that $w = \alpha^{\pm} w'$ or $w = \beta^{\pm} w'$ where $w'(1, 0, 0) = (a', b'\sqrt{2}, c')/3^{k-1}$. So

$$w(1, 0, 0) = (a, b\sqrt{2}, c)/3^k$$

where $a = a' \mp 4b', b = b' \pm 2a', c = 3c'$, or $a = 3a', b = b' \mp 2c', c = c' \pm 4b'$ depending on w beginning with $\alpha^{\pm}$ or $\beta^{\pm}$. It follows that a, b, c are always integers. We can also see that a word starting with α, when applied to $(1, 0, 0)$, will have its c divisible by 3 and a word starting with β, when applied to $(1, 0, 0)$, will have its a divisible by 3.

It remains to show that b never becomes divisible by 3. We will consider four cases depending on the first two elements of w: $\alpha^{\pm}\beta^{\pm}v$, $\beta^{\pm}\alpha^{\pm}v$, $\alpha^{\pm}\alpha^{\pm}v$ or $\beta^{\pm}\beta^{\pm}v$, where v can be the empty word. In the first case, since α is applied to a word starting with β, we will have $b = b' \pm 2a'$, and we know that a' is divisible by 3, so if b' is not divisible by 3, neither is b. In the second case, we know that c' is divisible by 3 and we are applying β so $b = b' \mp 2c'$ and b will not be divisible by 3 if b' was not. In the third case, we know that $b = b' \pm 2a'$, but $a' = a'' \mp 4b''$, so $b = b' \pm 2(a'' \mp 4b'') = b' \pm 2a'' - 8b'' = b' + (b'' \pm 2a'') - 9b'' = 2b' - 9b''$. Since b' is not divisible by 3, neither is b. In the forth case, we have $b = b' \mp 2c' = b' \mp 2(c'' \pm 4b'') = b' \mp 2c'' - 8b'' = b' - 8b'' - b'' + (b'' \mp 2c'') = 2b' - 9b''$ so b is not divisible by 3 if b' was not. $\square$

This subgroup L is free. Its elements are rotations whose axis contain the origin of $\mathbb{R}^3$. Each axis crosses the sphere twice. Let D be the set that contains all the points which are in the intersection between axes of elements from L with S^2. D is a countable subset of S^2, as L is countable. Given any point in D, there exists an element from L which fixes it.

Now, let's consider the set $S^2 \backslash D$. L acts on $S^2 \backslash D$ without fixed points. So we can transfer the paradoxical decomposition of L into $S^2 \backslash D$. The action of L in $S^2 \backslash D$ gives a partition of $S^2 \backslash D$ into orbits. There are uncountably many orbits. Each orbit is countable though, since L is. We choose a point in each orbit and call M the set of these points. By construction, the union of the orbits of M by L recovers the whole $S^2 \backslash D$. Now, if A_1, A_2, B_1 and B_2 are the sets of the paradoxical decomposition of L, then the sets A_1M, A_2M, B_1M and B_2M form a paradoxical decomposition of $S^2 \backslash D$. We have proved:

PROPOSITION 4.3. *There exists a countable subset D of S^2 such that $S^2 \backslash D$ is $SO(3)$-paradoxical.*

We could not have proved it without the **Axiom of Choice**. In fact, in the ZF axioms system can not be proved.

PROPOSITION 4.4. *S^2 and $S^2 \backslash D$ are $SO(3)$-equidecomposable.*

PROOF. We will find an element $\rho \in SO(3)$ such that D, $\rho(D)$, $\rho^2(D)$, ... , $\rho^n(D)$ have no intersection. As D is countable, we can find a line r that passes through the origin such that its two points of intersection with S^2 do not belong to D.

Now we consider the set of angles A. An angle θ belongs to A if there exists an n such that the rotation with axis r and angle $n\theta$ sends a point in D to another one.

As D is countable, we can find an angle ϕ that does not belong to A. Choosing the rotation with axis r and angle ϕ as ρ, we define:

$$\bar{D} = \bigcup_{n=0}^{\infty} \rho^n(D)$$

Here we have the equidecomposition:

$$S^2 = (S^2 \backslash \bar{D}) \cup \bar{D} \qquad S^2 \backslash D = (S^2 \backslash \bar{D}) \cup \rho(\bar{D})$$

So we have seen that S^2 is $SO(3)$-paradoxical. □

Extending these sets in a radial way, without the origin, it is easy to see that $B^3 \backslash O$ admits a similar paradoxical decomposition (as well as $\mathbb{R}^3 \backslash O$). Where O is the origin.

PROPOSITION 4.5. *B^3 and $B^3 \backslash O$ are equidecomposable.*

PROOF. As we did before, we need an element σ of the group G_3 that sends the origin to other points of the ball. We choose a rotation with an axis that passes really close to the origin and an angle which is an irrational multiple of π.

$$\bar{O} = \bigcup_{n=0}^{\infty} \sigma^n(O)$$

The equidecomposition is:

$$B^3 = (B^3 \backslash \bar{O}) \cup \bar{O} \qquad B^3 \backslash O = (B^3 \backslash \bar{O}) \cup \sigma(\bar{O})$$

□

Chapter 5
Følner characterization

Our main objective now is to find out which groups are amenable. It is easy to see that the finite groups are always amenable. If G is a group and A is a subgroup, we can define a finitely additive, G-invariant measure on $\mathcal{P}(G)$ with $\mu(G) = 1$:

$$\mu(A) = \frac{\#A}{\#G}$$

So we will focus on the infinite groups. It is natural to start with the most simple one, $\mathbb{Z}$, and that is why we need Følner characterization. We will try to define a finitely additive, $\mathbb{Z}$-invariant measure. To do so, we will use certain sets that will be very important and in those sets is where the Følner's idea lies.

Proposition 5.1. *$\mathbb{Z}$ is amenable.*

Proof. Let's consider the subset $C_n = [-n, n]$ which has $2n + 1$ elements. Let's define μ_n as follows:

$$\mu_n(A) = \frac{\#(A \cap C_n)}{\#C_n}$$

It is well defined over $\mathcal{P}(\mathbb{Z})$ and it is additive, but is it invariant under the action of the group $\mathbb{Z}$?

The generator of $\mathbb{Z}$ is 1, so gA would be $A+1=\{a+1\|a\in A\}$. If $A\cap C_n$ had k elements, then $(A+1)\cap C_n$ can have $k-1, k$ or $k+1$, depending on whether the numbers n and $-n-1$ belong to A.

$$|\mu_n(A+1)-\mu_n(A)| = \left|\frac{\#((A+1)\cap C_n)-\#(A\cap C_n)}{\#C_n}\right| \leq$$
$$\leq \frac{1}{2n+1}$$

It would be a good idea to define the measure in $\mathbb{Z}$ as

$$\mu(A)=\lim_{x\to\infty}\mu_n(A)$$

But, in general, this limit does not exist.

Even if this limit does not exist, a measure can be defined in the measures space. Consider now the space $[0,1]^{\mathcal{P}(\mathbb{Z})}$. Its elements are the applications from $\mathcal{P}(\mathbb{Z})$ to $[0,1]$. This space is a cartesian product of $\mathcal{P}(\mathbb{Z})$ times $[0,1]$, which is a compact set, so it is compact because of Tychonoff's theorem.

In this space, the subset of those measures that satisfy the properties of additivity and invariance is a closed set. Inside this closed set, $\mathcal{M}_n$ those measures that satisfy

$$|\nu(A+1)-\nu(A)|\leq\frac{1}{n}$$

is a closed set as well. If $n<m$ then $\mathcal{M}_n\supset\mathcal{M}_m$. Each $\mathcal{M}_n$ is non-empty, since the measures μ_n that we have defined belongs to it, so

$$\mathcal{M}=\bigcap_{n>0}\mathcal{M}_n\neq\emptyset$$

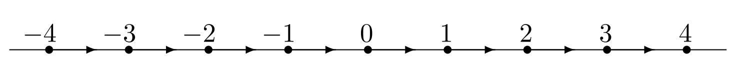
If a measure μ belongs to $\mathcal{M}$ (so it belongs to every $\mathcal{M}_n$), the difference within $\mu(A+1)$ and $\mu(A)$ must be 0, so this measure is invariant under the action of the group [**9**, pp. 191-192]. $\square$

Følner's idea was finding appropriate finite subsets of the group such that the measure in the group could be approximated by measures in these sets. As we have seen before, the finite sets of $\mathbb{Z}$ do not change much when an element of G acts on them. That happens because the cardinality of the boundary of those sets is little compared with the cardinality of the whole sets. But, what is the boundary of a subset of a group?

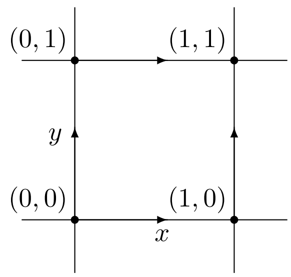
We need to introduce the concept of the **Cayley graph** for a finitely generated group.

Definition 5.2. Let G be a group and X be a finite set of G generators. The Cayley graph $\Gamma(G, X)$ of G respect to X is a graph which vertices are the elements of G and which edges are the triples (g, x, g') with $g, g' \in G$ and $x \in X$ such that $g' = gx$.

So, from every element g of the group, there is an edge $x \in X$ to the element gx. For example, in the Cayley graph of $\mathbb{Z}$ with the generators set $X = \{1\}$ is a line. The integer numbers are the vertices and an arise goes from n to $n+1$.



For example, $\mathbb{Z}^2$ is generated by two elements $x = (1, 0)$ and $y = (0, 1)$ such that $x + y = y + x$. Its Cayley graph is the net of all the integer points in $\mathbb{R}^2$ and its edges are segments of length 1, the horizontal are labeled with x, and vertical with y.



The Cayley graph of the free group F_2 is a tree, because there are not any relation. Let a and b be the two generators. From each vertex w there are two edges that goes to wa and wb. There are also two edges that arrives to w from wa^{-1} and wb^{-1}. The degree of every vertex is 4 so it is a regular tree.

Once we have the Cayley graph of the group G respect to X, we can define the boundary of a finite subset. Let C be a finite subset of G. We define the boundary of C, ∂C, as follows:

$$\partial C = \{g \in C | \text{there exists} \quad x \in X \quad \text{such that} \quad gx \notin C\}$$

We can see that the cardinality of the boundary of the sets C_n in the Cayley graph of $\mathbb{Z}$, is $\{-n, n\}$. In $\mathbb{Z}^2$, taking a square $R^2 = (\pm n, \pm n)$, its boundary has the elements with, at least, one of its coordinates equal to n in absolute value.

A set boundary is important because it is a measure of the way the set changes when a generator acts on the set. Taking a subset $C \subset G$, let's consider xC, where x is a generator and let's observe the symmetric difference within C and xC.

PROPOSITION 5.3. *Let C be a subset of G and x be a generator of G. The symmetric difference within C and xC are those elements which belongs to C but do not belong to xC or vice versa. All the elements in the symmetric difference within C and xC are in the boundary of C, or were in the boundary of C before multiplying them by x.*

PROOF. The elements of xC that do not belong to C look like xw and those w are in the boundary because if we multiply them by x, they go out from C. On the other hand, the elements of C that do not belong to xC are in the boundary of C as well, because if $w \notin xC$, then $x^{-1}w \notin C$. □

COROLLARY. *The symmetric difference within C and xC has fewer elements than the boundary of C.*

Our objective now is to find a sequence of sets C_n like the one defined above in the case of $\mathbb{Z}$, to define a measure μ_n like we did. Let A be an arbitrary subset of G

$$\mu_n(A) = \frac{\#A \cap C_n}{\#C_n}$$

So,

$$|\mu_n(xA) - \mu_n(A)| \leq \frac{|\#(xA \cap C_n) - \#(A \cap C_n)|}{\#C_n} \leq$$

$$\leq \frac{\#(C_n \Delta x C_n)}{\#C_n} \leq \frac{\#\partial C_n}{\#C_n}$$

This can be done with any family of finite subsets C_n in G. We need, the resulting measure (after applying Tychonoff) to be invariant. So we need the quotient between the cardinality of the boundary and the set tend to zero.

Definition 5.4. Let G be a finitely generated group. A **Følner's sequence** is a sequence of finite subsets $C_n \subset G$ such that

$$\lim_{n\to\infty} \frac{\#\partial C_n}{\#C_n} = 0$$

For example, $\mathbb{Z}^k$ is amenable. $C_n = [-n, n]^k$ and $\#\partial C_n$ is a multiple of n^{k-1}

Since we have seen that F_2 is not amenable, it does not admit any Følner sequence. If we consider its Cayley graph, a infinite regular tree, a ball can be defined by fixing a point w and a distance n. An element is in the ball if its distance to the fixed point is at most n. We can see

$$\#B_n(w) = 4 \cdot 3^n + 1$$
$$\#\partial B_n(w) = 4 \cdot 3^n - 4 \cdot 3^{n-1} = 4 \cdot 3^{n-1} \cdot (3 - 1)$$

and as we knew,

$$\lim_{n\to\infty} \frac{\#\partial B_n(w)}{\#B_n(w)} = \frac{4 \cdot 3^n + 1}{8 \cdot 3^{n-1}} = \frac{2}{3}$$

The balls cannot be the Følner sequence, in fact, we can see that any set family could work,

Proposition 5.5. *Any finite subset $A \subset F_2$ will fulfill*

$$\frac{\#\partial A}{\#A} \geq \frac{2}{3}$$

Proof. Let's consider an arbitrary finite subset $A \subset F_2$. We can suppose it is connected without loss of generality (if $\frac{a}{b}$ and $\frac{c}{d}$ are greater than $\frac{2}{3}$, then $\frac{a+c}{b+d}$ will be as well) and we will distinguish the vertexes depending on how many edges do they have. So we will have four sets of vertexes v_1, v_2, v_3 and v_4. Let V be the number of vertexes in A and let E be the number of edges in A, we know that $V = E + 1$ and:

$$V = v_1 + v_2 + v_3 + v_4$$
$$2 \cdot E = v_1 + 2 \cdot v_2 + 3 \cdot v_3 + 4 \cdot v_4$$

Multiplying the first expression by 4 and subtracting the second one, we get:

$$4 \cdot V - 2 \cdot E = 3 \cdot v_1 + 2 \cdot v_2 + v_3$$
$$4 \cdot V - 2 \cdot (V - 1) = 3 \cdot v_1 + 2 \cdot v_2 + v_3$$
$$2 \cdot (V + 1) = 3 \cdot v_1 + 2 \cdot v_2 + v_3$$
$$V = \frac{3}{2} \cdot v_1 + v_2 + \frac{1}{2} \cdot v_3 - 1$$

Since those vertexes that have less than 4 edges are in the boundary,

$$v_1 + v_2 + v_3 = \#\partial A$$

We have:

$$\frac{\#\partial A}{\#A} = \frac{v_1 + v_2 + v_3}{\frac{3}{2} \cdot v_1 + v_2 + \frac{1}{2} \cdot v_3 - 1} \geq \frac{v_1 + v_2 + v_3}{\frac{3}{2} \cdot (v_1 + \cdot v_2 + v_3)} = \frac{2}{3}$$

The last inequality used is true if and only if $\frac{1}{2} \cdot v_2 + v_3 \geq -1$, which is true since $v_i \geq 0$, for all i.

□

We have seen that those groups that bear a Følner's sequence are those such that the measure of a certain subset do not vary much when an element of the group is applied to that subset, and that is what Følner's Condition requires. A group G satisfies Følner's Condition if for any finite subset W of G and every $\epsilon > 0$, there is another finite subset W^* of G such that for any $g \in W$, $|gW^* \bigtriangleup W^*|/|W^*| \leq \epsilon$.

Our characterization of amenability with these Følner's sequence is useful in finitely generated groups. We will prove the next equivalence and the new characterization will help with non finitely generated groups.

PROPOSITION 5.6. *Let G be a group. There exists a Følner's sequence in G if and only if G satisfies Følner's Condition.*

PROOF. $\Rightarrow$

First we need a definition

DEFINITION 5.7. Let G be a group, $A \in G$ be a subset and k an integer. If $S = \{s_1, ..., s_m\}$ is a basis of G, we will call the **k-boundary**:

$$\partial_k A = \{g \in A | h_1, ..., h_m \in S such that h_{i_1} ... h_{i_k} g \notin A\}$$

We claim that if C_n are Følner's sets and

$$\lim_{n\to\infty} \frac{|\partial C_n|}{|C_n|} = 0$$

then

$$\lim_{n\to\infty} \frac{|\partial_k C_n|}{|C_n|} = 0$$

We will first relate the cardinality of the k-boundary and the boundary $|\partial_k A| \leq |\partial A| + m|\partial A| + m^2|\partial A| + ... + m^{k-1}|\partial A|$, so

$$\lim_{n\to\infty} \frac{|\partial A|}{\partial_k A} \geq \frac{|\partial A|}{|\partial A|(1 + m + ... + m^{k-1})} > 0$$

We have seen that for a finitely generated group and a finite integer k, the quotient $\frac{|\partial A|}{\partial_k A}$ is greater than zero. Since we know that $\lim_{n\to\infty} \frac{|\partial C_n|}{|C_n|} = 0$, we can rewrite:

$$\lim_{n\to\infty} \frac{|\partial C_n|}{|C_n|} = \lim_{n\to\infty} \frac{|\partial C_n|}{|\partial_k C_n|} \frac{|\partial_k C_n|}{|C_n|} = 0$$

So we can see that a Følner's sequence will satisfy

$$\lim_{n\to\infty} \frac{|\partial_k C_n|}{|C_n|} = 0 \quad \text{for any} \quad k.$$

With this, given a set $W \in G$, a real number $\epsilon > 0$ and an element $g \in W$ we want to find the set $W*$ such that meet the Følner's Condition. We know that the symmetric difference between a set A and gA for any $g \in S$ is smaller than ∂A. It happens the same if $g \in G$ with the k-boundary. The proof is the same. Now, we consider the Følner's sequence and choose a C_n such that meet $\frac{|\partial_k C_n|}{|C_n|} \leq \epsilon$. There exists an N big enough to meet it because that quotient goes to zero. We take that C_N as W^* and

$$\frac{|gW^* \triangle W^*|}{|W^*|} \leq \frac{|\partial_k W^*|}{|W^*|} \leq \epsilon$$

$\Leftarrow$ Given any subset $W \subset G$, any element $g \in G$ and any real number $\epsilon > 0$, there exists a subset $W^* \subset G$ such that

$$\frac{|gW^* \triangle W^*|}{|W^*|} \leq \epsilon$$

We know that that symmetric difference is smaller than the k-boundary (where k is the number of generators if we decompose g), but if we let m be the cardinality of the basis,

$$\frac{|gW * \triangle W *|}{|W *|} \leq \frac{|\partial_k W *|}{|W *|} \leq \frac{m|gW * \triangle W *|}{|W *|}$$

For any $n \in \mathbb{N}$ we will give ϵ the value $\frac{m}{n}$. For any n we will take as C_n the W^* that meets the previous equation.

$\square$

THEOREM 5.8. *A finitely generated group is amenable if and only if it satisfies Følner's Condition.*

That a finitely generated group being amenable implies the existence of a Følner's sequence, or that the Følner's Condition is verified is a tough proof that we will omit in order to not extend this project talking about concepts not very related to the central topic. For instance, we will give a proof of the other direction [**1**, p. 161]

PROOF. $\Leftarrow$)

For each finite $W \subseteq G$ and $\epsilon > 0$, let $\mathcal{M}_{W,\epsilon}$ consist of those finitely additive functions $\mu : \mathcal{P}(G) \to [0,1]$ such that $\mu(G) = 1$, and for each $g \in W$ and $A \subseteq G$, $|\mu(A) = \mu(gA)| \leq \epsilon$. Then $\mathcal{M}_{W,\epsilon}$ is a closed subset of $[0,1]^{\mathcal{P}(G)}$, and, as usual, a compactness argument will complete the proof once it is shown that each $\mathcal{M}_{W,\epsilon}$, is nonempty. For this, simply define $\mu(A)$ to be $|A \cap W^*|/|W^*|$ where W^* is as provided by Følner's Condition; then μ is in $\mathcal{M}_{W,\epsilon}$. $\square$

Chapter 6
Amenability

One of the main aims of this dissertation is to get to understand the concept of **amenability**. First, we will give several characterizations that are equivalent and we have already seen (the interested reader can refer to [**1**, p. 157] for other characterizations).

THEOREM 6.1. *For a group G, the following are equivalent:*

(1) G is amenable.
(2) There is a left-invariant mean on G.
(3) G is not paradoxical.
(4) G satisfies Følner's Condition: For any finite subset W of G and every $\epsilon > 0$, there is another finite subset W^* of G such that for any $g \in W$, $|gW^* \triangle W^*|/|W^*| \leq \epsilon$.

We have already proved that (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (2). The conversely of these are proved using topological methods, so we will not include them in this project.

The other main goal is to get to know which groups are amenable and which ones do not. Now that we know what means being amenable, we can try to find out which groups have this property. Following theorems will help to get to know which groups are in the amenable groups class (AG from now on). In the next section we will prove that the

Banach-Tarski paradox is not possible in the plane. To do it, we will prove that the isometry group on the plane is amenable [**1**, p. 149]. We will need several results:

PROPOSITION 6.2. *If G is the direct union of a directed system $\{G_\alpha : \alpha \in I\}$ of amenable groups, then G is amenable.*

PROOF. We have $G = \cup\{G_\alpha : \alpha \in I\}$, where each G_α is amenable, so it bears a measure μ_α. Since it is a directed system, for each α, $\beta \in I$, there is some $\gamma \in I$ such that G_α and G_β are both subgroups of G_γ.

We will consider the space $[0,1]^{\mathcal{P}(G)}$. For each α, let $\mathcal{M}_\alpha$ consist of those finitely additive measures $\mu : \mathcal{P}(G) \to [0,1]$ such that $\mu(G) = 1$ and $\mu(gA) = \mu(A)$ whenever $g \in G_\alpha$. Every G_α is nonempty, since we can define $\mu(A) = \mu_\alpha(A \cap G_\alpha)$. $\mathcal{M}_\alpha$ is closed because if a measure that fails to be in it, this failure is evident from finitely many values of the function. We will intersect all this measures in order to prove that the limit exist. Since G_α, $G_\beta \subseteq G_\gamma$, the measures such that $\mu(gA) = \mu(A)$ whenever $g \in G_\gamma$ are included in $\mathcal{M}_\alpha$ and in $\mathcal{M}_\beta$ as well. So $\mathcal{M}_\gamma \subseteq \mathcal{M}_\alpha \cap \mathcal{M}_\beta$. We have seen that the collection $\mathcal{M}_\alpha : \alpha \in I$ has the finite intersection property. Then, by compactness, there is some $\mu \in \cap \mathcal{M}_\alpha : \alpha \in I$ and such μ witnesses the amenability of G. ☐

THEOREM 6.3. *Abelian groups are amenable.*

PROOF. Any group is the direct union of its finitely generated subgroups, so it suffices to consider finitely generated abelian groups since we have proved in the previous proposition that this will imply the amenability of any abelian group, finitely generated or not.

Let G a finitely generated abelian group and $g_1, ..., n_n$ be the generating set. We will show that for all $\epsilon > 0$, there is a function $\mu_\epsilon : \mathcal{P}(G) \to [0,1]$ such that

(1) $\mu_\epsilon(G) = 1$
(2) μ_ϵ is finitely additive
(3) μ_ϵ is almost invariant with respect to the generators in the sense that for each $A \subseteq G$ and generator g_k, $|\mu_\epsilon(A) - \mu_\epsilon(g_k A)| \leq \epsilon$.

To prove the existence of such a measure, we will first consider the case where G has a single generator g_1. Choose N large enough such that $2/N \leq \epsilon$, and let's define:

$$\mu_\epsilon(A) = |\{i : 1 \leq i \leq N \text{ and } g_1^i \in A\}|/N$$

It is easy to see that $\mu_\epsilon(A)$ differs from $\mu_\epsilon(g_1 A)$ by no more than $2/n \leq \epsilon$. For the general case, with m generators, we will choose N as before, but now we will define:

$$\begin{gathered}\mu_\epsilon(A) = |\{(i_1, ..., i_m) : 1 \leq i_1, ..., i_m \leq N \\ \text{and } g_1^{i_1} ... g_m^{i_m} \in A\}|/N^m\end{gathered}$$

Then $\mu_\epsilon(G) = 1$, μ is finitely additive. Let's consider the case where we want to calculate how much differs $\mu_\epsilon(A)$ from $\mu_\epsilon(g_k A)$. Since g_k commutes with the other generators, the vector $(i_1, ..., i_m)$ will only change in its k coordinate, so the difference will be at most:

$$\begin{gathered}|\{(i_1, ..., i_m) : 1 \leq i_1, ..., i_{k-1}, i_{k+1}, ..., i_m \leq N \\ \text{and } i_k = 1 \text{ or } i_k = N+1\}|/N^m = \\ = 2N^{m-1}/N^m = 2/N \leq \epsilon\end{gathered}$$

as desired.

Now that we have proved the existence of this almost invariant measure, we may let $\mathcal{M}_\epsilon$ denote the set of functions from $\mathcal{P}(G)$ to $[0,1]$ satisfying conditions (1), (2) and (3). Then, each $\mathcal{M}_\epsilon$ is nonempty, since we have proved the existence of at least one measure there, and it is closed, since if a function fails to lie in $\mathcal{M}_\epsilon$, it fails from finitely many values of the function (as in the proof of the previous proposition). The collection of the sets $\mathcal{M}_\epsilon$ has the finite intersection property: $\bigcap \mathcal{M}_{\epsilon_i} = \mathcal{M}_{min \epsilon_i}$, which is nonempty. We can conclude, by the compactness of $[0,1]^{\mathcal{P}(G)}$ that there is some μ lying in that intersection. Such a measure is left-invariant with respect to each generator, and hence, with respect to any member of G. $\square$

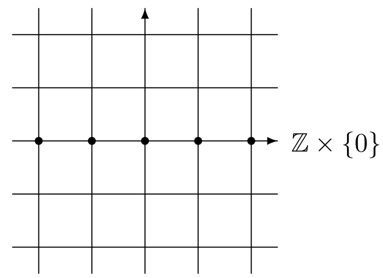
PROPOSITION 6.4. *A subgroup of an amenable group is amenable.*

PROOF. Let μ be the measure on G, and suppose H is a subgroup of G. The measure that first comes to mind would be:

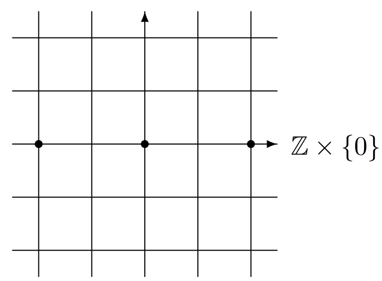
$$\mu_H(A) = \frac{\mu(A)}{\mu(H)}$$

the problem with this measure are the cases where $\mu(H)$ is zero. In those cases, let M be a set of representatives (choice set) for the collection of right cosets of H in G. Then define ν on $\mathcal{P}(H)$ by $\nu(A) = \mu(\bigcup\{Ag : g \in M\})$. It is easy to check that ν is a measure on H.

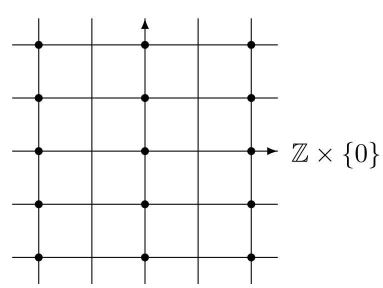
The idea behind this construction is to give to a certain $A \subseteq H$ the measure that would correspond to a subset of G that represents the same proportion in G as the represented by A in H. The easiest way to explain in is with an example. Let's consider $\mathbb{Z}^2$ and will consider $\mathbb{Z} \times \{0\}$ as H. Let μ be the measure in $\mathbb{Z}^2$. It happens that $\mu(H) = 0$, so for any $A \subseteq H$, $\mu(A) = 0$.



Now, let's consider the subset of the even numbers:



Let's pay attention to the coset $\mathbb{Z}\times\{1\}$. There are infinitely many elements $g \in \mathbb{Z}^2$ such that $gA = \mathbb{Z}\times\{1\}$, in fact, any $(i, 1)$ with $i \in \mathbb{Z}$ would meet $(i, 1) \times A = \mathbb{Z} \times \{1\}$. That is why we choose a set of representatives. In this case, we will consider the set $M = \{(0, i) : i \in \mathbb{Z}\}$. Now, the set $\cup\{Ag : g \in M\}$ is:



and $\mu(\cup\{Ag : g \in M\}) = \frac{1}{2}$. That is the measure we wanted the even numbers to have in the subset H. $\square$

From propositions 6.2 and 6.4, we can rewrite:

COROLLARY. *A group is amenable if and only if all of its finitely generated subgroups are.*

Now that we know that abelian groups are amenable, if we combine that with the following proposition, we will be able to say that all solvable groups are amenable.

PROPOSITION 6.5. *If N is a normal subgroup of G, and each of N, $G\backslash N$ are amenable, then G is amenable.*

PROOF. Let ν_1, ν_2 be measures on N, G/N, respectively. For any $A \subseteq G$ let $f_A : G \to \mathbb{R}$ be defined by $f_A(g) = \nu_1(N \cap g^{-1}A)$. Then, for those elements of the set g_1, g_2 that define the same coset of N in G, $f_A(g_1) = f_A(g_2)$. If they define the same coset, then $g_1 = g_2h \in N$.

$$f_A(g_2) = \nu_1(N \cap g_2^{-1}A) = \nu_1(N \cap hg_1^{-1}A) =$$
$$= \nu_1(h(N \cap g_1^{-1}A)) = \nu_1(N \cap g_1^{-1}A) = f_A(g_1)$$

So, f_A can be regarded as a bounded real-valued function with domain G/N. Now, let's define $\mu(A)$ to be $\int f_A d\nu_2$. Since $f_G = \chi_G$, $\mu(G) = 1$, and if A, $B \subseteq G$, $A \cap B = \emptyset$, then for any $g \in G$, $g^{-1}A \cap g^{-1}B = \emptyset$, so $f_{A\cup B}(g) = f_A(g) + f_B(g)$. This yields the finite additivity of μ. Finally,

$$f_{gA}(g_0) = \nu_1(N \cap g_0^{-1}A) = f_A(g^{-1}g_0) =_g (f_A)(g_0)$$

so, the left-invariance of the integral yields that $\mu(gA) = \int f_{gA} d\nu_2 = \int_g (f_A) d\nu_2 = \int (f_A) d\nu_2 = \mu(A)$.

$\square$

DEFINITION 6.6. Let G be a finite group. G is solvable if there exists a finite sequence of subgroups $\{G_i\}_{i=1}^{n} \subset G$ such that:

$$\{1_G\} = G_0 \subseteq G_1 \subseteq ... \subseteq G_n = G$$

where for each $i = 0, 1, ..., n-1$, G_i is a normal subgroup of G_{i+1} and every quotient G_{i+1}/G_i is abelian.

We have recalled this definition because as a consequence of proposition theorem 6.3 and proposition 6.5, it follows that:

COROLLARY. *Every solvable group is amenable.*

Let's recall as well the definition of EG, the class of the elementary groups [**1**, p. 12]. This is the smallest class containing all finite groups and all abelian groups, and satisfying the following properties:

(i) if H is a subgroup of $G \in EG$, then $H \in EG$.
(ii) if H is a normal subgroup of $G \in EG$, then $G/H \in EG$.
(iii) if H is a normal subgroup of G, and both H and G/H are in EG, then $G \in EG$.
(iv) if $\{G_i : i \in I\}$ is a directed system with respect to the subgroup relation, and each $G_i \in EG$, then the union of the G_i is a group in EG.

Condition (iv), in the presence of (i) is equivalent to if all finitely generated subgroups of G are in EG, then $G \in EG$. It is immediate consequence of the previous theorems that all elementary groups are amenable, and therefore $EG \subseteq AG$. The only condition that we have not proved yet is $[ii]$:

PROPOSITION 6.7. *If N is a normal subgroup of the amenable group G, then $G \backslash N$ is amenable*

PROOF. If μ is a measure on G, then define $\nu : \mathcal{P}(G/N) \to [0,1]$ by setting $\nu(A) = \mu(\cup\{aN : a \in A\})$. Again, it is a routine to check that ν is as desired. $\square$

We denote by NF the class of the groups not containing a free group of rank 2. We can write: $EG \subseteq AG \subseteq NF$. But, are those proper inclusions? Do non amenable groups non containing a free group of rank 2 exist? Do non elementary amenable groups exist?

S. I. Adyan showed [**7**] in 1968 that the Burnside group $B(2,665)$ belongs to NF but not to EG, so there was an example clarifying that at least one of the inclusions is proper. Rotislav Ivanovic Grigorchuk gave a counterexample in 1984 to the Milnor-Wolf Conjecture [**6**, pp. 939-985], showing that the first inclusion is proper. A. Y. Ol'Shanskii, in 1980 proved that the conjecture $AG = NF$ is false [**8**, pp. 180-181].

Chapter 7
Isometry group of $\mathbb{R}^2$

INTRO

In this section we are going to see that the isometry group in $\mathbb{R}^2$ is amenable. By proving this, we will show that the paradox is not possible in $\mathbb{R}^2$. In order to prove it, we will rewrite proposition 6.4, 6.5 and 6.7 as follows:

PROPOSITION 7.1. *Let G be a group and $H \lhd G$ be a normal subgroup. If we have this exact sequence:*

$$1 \to H \to G \to G/H \to 1$$

then G is amenable if and only if H and G/H are amenable.

THEOREM 7.2. *The isometry group G_2 of $\mathbb{R}^2$ is amenable.*

PROOF. It is well known that G_2 has a subgroup G_2^+ of index 2 that contains the isometries that preserve the orientation. Those isometries are formed with a rotation (an element of S^1) and a translation (an element of $\mathbb{R}^2$), so we have two exact sequences:

$$1 \to G_2^+ \to G_2 \to \mathbb{Z}/2\mathbb{Z} \to 1$$
$$1 \to \mathbb{R}^2 \to G_2^+ \to S^1 \to 1$$

Since $\mathbb{R}^2$ and S^1 are abelian, they are amenable, and so it is G_2^+. $\mathbb{Z}/2\mathbb{Z}$ is finite, so it is amenable, so we see that G_2 is amenable. $\square$

With this last statement, we conclude that there exist a measure defined in $\mathcal{P}(\mathbb{R}^2)$, finitely additive, G_2-invariant and such that the measure of the whole $\mathbb{R}^2$ is 1.

Chapter 8
Growth of groups

In this section we will talk about the growth of a group. This can be understood as the way the balls increase their volume in a group or, in other words, how does the set of words of length n increase when n increases. It will be interesting to study this concept because sometimes we will be able to determine whether a group is amenable or not in terms of growth [**5**, pp. 2-3].

Let $S = \{s_1, ..., s_k\}$ be a generating set of a group $G =< S >$. For every group element $g \in G$, denote by $l(g) = l_S(g)$ the length of the shortest decomposition $g = s_{i_1}^{\pm 1} \cdots s_{i_l}^{\pm 1}$. Let $\gamma_G^S(n)$ be the number of elements $g \in G$ such that $l(g) \leq n$.

DEFINITION 8.1. Function $\gamma = \gamma_G^S$ is called the **growth function** of the group G with respect to the generating set S.

Clearly, $\gamma(n) \leq \Sigma_{i=0}^{n} (2k)^n \leq (2k+1)^n$.

We call a function $f : \mathbb{N} \to \mathbb{R}$ **polynomial** if $f(n) \sim n^\alpha$, for some $\alpha > 0$. A function f is called **superpolynomial** if there exists a limit $\frac{ln\gamma(n)}{lnn} \to \infty$ as $n \to \infty$. For example, n^k for $k \in \mathbb{R}$ is polynomial while n^n is superpolynomial. In the same way, a function f will be called **exponential** if $f(n) \sim e^n$. A function f is called **subexponential** if there exists a limit $\frac{ln\gamma(n)}{n} \to 0$ as $n \to \infty$. Functions like k^n for any $k \in \mathbb{R}$ are exponential, but n^k is subexponential.

We must remark that there are functions that can not be categorized on the previous categories. Function $e^{sin(n)}$ fluctuates between 1 and e^n, so it is neither polynomial nor superpolynomial, neither exponential nor subexponential.

DEFINITION 8.2. A function f is said to have **intermediate growth** if f is both subexponential and superpolynomial.

For example, $n^{log(log(n))}$ has intermediate growth. Elementary groups have subexponential growth, and groups with intermediate growth are always amenable, so there exists a class of groups between elementary groups and amenable groups that can be defined by adding groups with intermediate growth to the elementary ones.

Groups of subexponential growth are amenable. Groups with exponential growth can be either amenable or not. For example, we know that the free group is not amenable so we know for sure that its growth is exponential. In fact, if we define the sets B_n with $n \in \mathbb{N}$, we will see that $|B_n|$ grows exponentially. Each of those subsets will have more than 4^n elements.

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